Chapter review 8

1 $f(x) = 10x^2$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{10(x+h)^2-10x^2}{h}$$
$$= \lim_{h\to 0}\frac{10x^2+20xh+10h^2-10x^2}{h}$$
$$= \lim_{h\to 0}\frac{20xh+10h^2}{h}$$
$$= \lim_{h\to 0}\frac{h(20x+10h)}{h}$$
$$= \lim_{h\to 0}(20x+10h)$$

As $h \to 0$, $20x + 10h \to 20x$
So $f'(x) = 20x$

2 a A has coordinates (1, 4).
The y-coordinate of B is
$(1 + \delta x)^3 + 3(1 + \delta x)$
$= 1^3 + 3\delta x + 3(\delta x)^2 + (\delta x)^3 + 3 + 3\delta x$
$= (\delta x)^3 + 3(\delta x)^2 + 6\delta x + 4$
Gradient of AB

$$= \frac{y_2-y_1}{x_2-x_1}$$
$$= \frac{(\delta x)^3+3(\delta x)^2+6\delta x+4-4}{\delta x}$$
$$= \frac{(\delta x)^3+3(\delta x)^2+6\delta x}{\delta x}$$
$$= (\delta x)^2+3\delta x+6$$

b As $\delta x \to 0$, $(\delta x)^2 + 3\delta x + 6 \to 6$
Therefore, the gradient of the curve at point A is 6.

3 $y = 3x^2 + 3 + \dfrac{1}{x^2} = 3x^2 + 3 + x^{-2}$

$$\frac{dy}{dx} = 6x - 2x^{-3} = 6x - \frac{2}{x^3}$$

When $x=1$, $\dfrac{dy}{dx} = 6\times 1 - \dfrac{2}{1^3}$
$= 4$

When $x=2$, $\dfrac{dy}{dx} = 6\times 2 - \dfrac{2}{2^3}$
$= 12 - \dfrac{2}{8}$
$= 11\dfrac{3}{4}$

When $x=3$, $\dfrac{dy}{dx} = 6\times 3 - \dfrac{2}{3^3}$
$= 18 - \dfrac{2}{27}$
$= 17\dfrac{25}{27}$

The gradients at points A, B and C are 4, $11\frac{3}{4}$ and $17\frac{25}{27}$, respectively.

4 $y = 7x^2 - x^3$

$\dfrac{dy}{dx} = 14x - 3x^2$

$\dfrac{dy}{dx} = 16$ when

$14x - 3x^2 = 16$
$3x^2 - 14x + 16 = 0$
$(3x - 8)(x - 2) = 0$
$x = \dfrac{8}{3}$ or $x = 2$

5 $y = x^3 - 11x + 1$

$\dfrac{dy}{dx} = 3x^2 - 11$

$\dfrac{dy}{dx} = 1$ when

$3x^2 - 11 = 1$
$3x^2 = 12$
$x^2 = 4$
$x = \pm 2$

When $x = 2$, $y = 2^3 - 11(2) + 1 = -13$
When $x = -2$, $y = (-2)^3 - 11(-2) + 1 = 15$
The gradient is 1 at the points (2, −13) and (−2, 15).

6 a $f(x) = x + \dfrac{9}{x} = x + 9x^{-1}$

$f'(x) = 1 - 9x^{-2} = 1 - \dfrac{9}{x^2}$

6 b $f'(x) = 0$ when

$$\frac{9}{x^2} = 1$$
$$x^2 = 9$$
$$x = \pm 3$$

7 $y = 3\sqrt{x} - \dfrac{4}{\sqrt{x}} = 3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}} + 2x^{-\frac{3}{2}}$$

8 a $y = 12x^{\frac{1}{2}} - x^{\frac{3}{2}}$

$$\frac{dy}{dx} = 12\left(\frac{1}{2}\right)x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}}$$
$$= 6x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}}$$
$$= \frac{3}{2}x^{-\frac{1}{2}}(4 - x)$$

b The gradient is zero when $\dfrac{dy}{dx} = 0$:

$$\frac{3}{2}x^{-\frac{1}{2}}(4 - x) = 0$$

$x = 4$

When $x = 4$, $y = 12 \times 2 - 2^3 = 16$

The gradient is zero at the point with coordinates (4, 16).

9 a $\left(x^{\frac{3}{2}} - 1\right)\left(x^{-\frac{1}{2}} + 1\right) = x + x^{\frac{3}{2}} - x^{-\frac{1}{2}} - 1$

b $y = x + x^{\frac{3}{2}} - x^{-\frac{1}{2}} - 1$

$$\frac{dy}{dx} = 1 + \frac{3}{2}x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$$

c When $x = 4$, $\dfrac{dy}{dx} = 1 + \dfrac{3}{2} \times 2 + \dfrac{1}{2} \times \dfrac{1}{4^{\frac{3}{2}}}$

$$= 1 + 3 + \frac{1}{16}$$
$$= 4\frac{1}{16}$$

10 Let $y = 2x^3 + \sqrt{x} + \dfrac{x^2 + 2x}{x^2}$

$$= 2x^3 + x^{\frac{1}{2}} + \frac{x^2}{x^2} + \frac{2x}{x^2}$$
$$= 2x^3 + x^{\frac{1}{2}} + 1 + 2x^{-1}$$
$$\frac{dy}{dx} = 6x^2 + \frac{1}{2}x^{-\frac{1}{2}} - 2x^{-2}$$
$$= 6x^2 + \frac{1}{2\sqrt{x}} - \frac{2}{x^2}$$

11 The point (1, 2) lies on the curve with equation $y = ax^2 + bx + c$, so

$2 = a + b + c$ **(1)**

The point (2, 1) also lies on the curve, so

$1 = 4a + 2b + c$ **(2)**

(2) − **(1)** gives:

$-1 = 3a + b$ **(3)**

$$\frac{dy}{dx} = 2ax + b$$

The gradient of the curve is zero at (2, 1), so

$0 = 4a + b$ **(4)**

(4) − **(3)** gives:

$1 = a$

Substituting $a = 1$ into **(3)** gives $b = -4$

Substituting $a = 1$ and $b = -4$ into **(1)** gives $c = 5$

Therefore, $a = 1$, $b = -4$, $c = 5$

12 a $y = x^3 - 5x^2 + 5x + 2$

$$\frac{dy}{dx} = 3x^2 - 10x + 5$$

b i $\dfrac{dy}{dx} = 2$

$$3x^2 - 10x + 5 = 2$$
$$3x^2 - 10x + 3 = 0$$
$$(3x - 1)(x - 3) = 0$$
$$x = \frac{1}{3} \text{ or } 3$$

$x = 3$ is the coordinate at P,

so $x = \dfrac{1}{3}$ at Q.

12 b ii $x = 3 \Rightarrow y = 27 - 45 + 15 + 2 = -1$

So the equation of the tangent is

$y + 1 = 2(x - 3)$

$y = 2x - 7$

iii When $x = 0$, $y = -7$

and when $y = 0$, $x = \frac{7}{2}$

So points R and S are $(0, -7)$ and $(\frac{7}{2}, 0)$.

Length of $RS = \sqrt{(-7)^2 + (\frac{7}{2})^2}$

$= 7\sqrt{1 + \frac{1}{4}} = \frac{7}{2}\sqrt{5}$

13 $y = \frac{8}{x} - x + 3x^2 = 8x^{-1} - x + 3x^2$

$\frac{dy}{dx} = -8x^{-2} - 1 + 6x = -\frac{8}{x^2} - 1 + 6x$

When $x = 2$, $y = \frac{8}{2} - 2 + 3 \times 2^2 = 14$

$\frac{dy}{dx} = -\frac{8}{4} - 1 + 12 = 9$

The equation of the tangent through the point (2, 14) with gradient 9 is

$y - 14 = 9(x - 2)$

$y = 9x - 18 + 14$

$y = 9x - 4$

The normal at (2, 14) has gradient $-\frac{1}{9}$.

So its equation is

$y - 14 = -\frac{1}{9}(x - 2)$

$9y + x = 128$

14 a $2y = 3x^3 - 7x^2 + 4x$

$y = \frac{3}{2}x^3 - \frac{7}{2}x^2 + 2x$

$\frac{dy}{dx} = \frac{9}{2}x^2 - 7x + 2$

At (0, 0), $x = 0$, gradient of curve is $0 - 0 + 2 = 2$.

Gradient of normal at (0, 0) is $-\frac{1}{2}$.

The equation of the normal at (0, 0) is

$y = -\frac{1}{2}x$.

At (1, 0), $x = 1$, gradient of curve is

$\frac{9}{2} - 7 + 2 = -\frac{1}{2}$.

Gradient of normal at (1, 0) is 2.

14 a The equation of the normal at (1, 0) is $y = 2(x - 1)$.

The normals meet when $y = 2x - 2$ and $y = -\frac{1}{2}x$:

$2x - 2 = -\frac{1}{2}x$

$4x - 4 = -x$

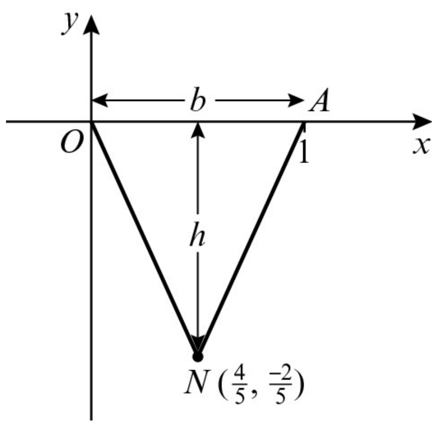
$5x = 4$

$x = \frac{4}{5}$

$y = 2\left(\frac{4}{5}\right) - 2 = -\frac{2}{5}$ $\left(\text{check in } y = -\frac{1}{2}x\right)$

N has coordinates $\left(\frac{4}{5}, -\frac{2}{5}\right)$.

b



Area of $\triangle OAN = \frac{1}{2}$ base × height

Base (b) = 1

Height (h) = $\frac{2}{5}$

Area = $\frac{1}{2} \times 1 \times \frac{2}{5} = \frac{1}{5}$

15 $y = x^3 - 2x^2 - 4x - 1$

When $x = 0$, $y = -1$ so the point P is $(0, -1)$.

For the gradient of line L:

$\frac{dy}{dx} = 3x^2 - 4x - 4$

At point P, when $x = 0$, $\frac{dy}{dx} = -4$

The y-intercept of line L is -1.

Equation of L is $y = -4x - 1$.

Point Q is where the curve and line intersect:

$x^3 - 2x^2 - 4x - 1 = -4x - 1$

$x^3 - 2x^2 = 0$

15 $x^2(x-2)=0$
$x=0$ or 2
$x=0$ at point P, so $x=2$ at point Q.
When $x=2$, $y=-9$ substituting into the original equation
Using Pythagoras' theorem:
$$\text{distance } PQ = \sqrt{(2-0)^2+(-9-(-1))^2} = \sqrt{68} = \sqrt{4\times 17} = 2\sqrt{17}$$

16 $y=x^3-6x^2+9x$
$\frac{dy}{dx}=3x^2-12x+9 \Rightarrow 3x^2-12x+9=0$
at a turning point.
$3x^2-12x+9=0 \Rightarrow x^2-4x+3=0$
$x^2-4x+3=(x-3)(x-1) \Rightarrow x=1, 3$
$\Rightarrow y=4, 0 \Rightarrow (1, 4)$ and $(3, 0)$

17 a $f(x)=200-\frac{250}{x}-x^2$
$f'(x)=\frac{250}{x^2}-2x$

b At the maximum point, B, $f'(x)=0$
$$\frac{250}{x^2}-2x=0$$
$$\frac{250}{x^2}=2x$$
$$250=2x^3$$
$$x^3=125$$
$$x=5$$
When $x=5$, $y=f(5)=200-\frac{250}{5}-5^2$
$=125$
The coordinates of B are $(5, 125)$.

18 a $OP^2=x^2+\left(5-\frac{1}{2}x^2\right)^2$
$\Rightarrow OP^2=x^2+25-5x^2+\frac{1}{4}x^4$
$=\frac{1}{4}x^4-4x^2+25$

b $f'(x)=x^3-8=0 \Rightarrow x(x^2-8)=0$
$\Rightarrow x=0,\ x=\pm\sqrt{8}$

c $x=0 \Rightarrow OP^2=25 \Rightarrow OP=5$
$x=\pm\sqrt{8} \Rightarrow OP^2=1 \Rightarrow OP=1$
(OP is a distance so must be positive)